Exercise 1D

1 **a** $\text{RHS} = 2\sinh A \cosh A$

$$= 2\left(\frac{e^{A} - e^{-A}}{2}\right)\left(\frac{e^{A} + e^{-A}}{2}\right)$$
$$= \frac{1}{2}\left(e^{2A} - 1 + 1 - e^{-2A}\right)$$
$$= \frac{e^{2A} - e^{-2A}}{2}$$
$$= \sinh 2A = \text{LHS}$$

b $\text{RHS} = \cosh A \cosh B - \sinh A \sinh B$

$$= \left(\frac{e^{A} + e^{-A}}{2}\right)\left(\frac{e^{B} + e^{-B}}{2}\right) - \left(\frac{e^{A} - e^{-A}}{2}\right)\left(\frac{e^{B} - e^{-B}}{2}\right)$$
$$= \frac{e^{A+B} + e^{-A+B} + e^{A-B} + e^{-A-B}}{4}$$
$$- \frac{e^{A+B} - e^{-A+B} - e^{A-B} + e^{-A-B}}{4}$$
$$= \frac{2\left(e^{-A+B} + e^{A-B}\right)}{4}$$
$$= \frac{e^{A-B} + e^{-(A-B)}}{2}$$
$$= \cosh(A - B) = \text{LHS}$$

c $\text{RHS} = 4\cosh^3 A - 3\cosh A$

$$= 4\left(\frac{e^{A} + e^{-A}}{2}\right)^3 - 3\left(\frac{e^{A} + e^{-A}}{2}\right)$$

$$\left(e^{A} + e^{-A}\right)^3 = e^{3A} + 3e^{2A}e^{-A} + 3e^{A}e^{-2A} + e^{-3A}$$
$$= e^{3A} + 3e^{A} + 3e^{-A} + e^{-3A}$$

Use the expansion $(x + y)^3 = x^3 + 3x^2 y + 3xy^2 +$

$$\text{RHS} = \frac{e^{3A} + 3e^{A} + 3e^{-A} + e^{-3A}}{2} - \frac{3(e^{A} + e^{-A})}{2}$$
$$= \frac{e^{3A} + e^{-3A}}{2}$$
$$= \cosh 3A = \text{LHS}$$

1 d
$$\text{RHS} = 2\sin\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$$
$$= 2\left(\frac{e^{\frac{A-B}{2}} - e^{\frac{-A+B}{2}}}{2}\right)\left(\frac{e^{\frac{A+B}{2}} + e^{\frac{-A-B}{2}}}{2}\right)$$
$$= \frac{1}{2}\left(e^{\frac{A-B}{2}+\frac{A+B}{2}} - e^{\frac{-A+B}{2}+\frac{A+B}{2}} + e^{\frac{A-B}{2}+\frac{A-B}{2}} - e^{\frac{-A+B}{2}+\frac{-A-B}{2}}\right)$$
$$= \frac{1}{2}\left(e^{A} - e^{B} + e^{-B} - e^{-A}\right)$$
$$= \frac{1}{2}\left(e^{A} - e^{-A}\right) - \frac{1}{2}\left(e^{B} - e^{-B}\right)$$
$$= \sinh A - \sinh B$$
$$= \text{LHS}$$

2 a
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\sinh(A-B) = \sinh A\cosh B - \cosh A\sinh B$$

Replace sin x by sinh x and cos x by cosh x.

b
$$\sin 3A = 3\sin A - 4\sin^3 A$$
$$= 3\sin A - 4\sin A\sin^2 A$$
$$\sinh 3A = 3\sinh A + 4\sinh^3 A$$

Replace $\sin^2 A$, the product of two sine terms, by $-\sinh^2 A$.

c
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cosh A + \cosh B = 2\cosh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$$

Replace cos x by cosh x.

d
$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$
$$\cosh 2A = \frac{1+\tanh^2 A}{1-\tanh^2 A}$$

$\tan^2 A = \dfrac{\sin^2 A}{\cos^2 A}$, so there is a product of two sines.

e
$$\cos 2A = \cos^4 A - \sin^4 A$$
$$= \cos^4 A - (\sin^2 A)(\sin^2 A)$$
$$\cosh 2A = \cosh^4 A - (-\sinh^2 A)(-\sinh^2 A)$$
$$= \cosh^4 A - \sinh^4 A$$

Replace

3 **a** Using $\cosh^2 x - \sinh^2 x = 1$

$4 - \sinh^2 x = 1$

$\sinh^2 x = 3$

$\sinh x = \pm\sqrt{3}$

Both positive and negative values of sinh x are possible.

b Using $\tanh x = \dfrac{\sinh x}{\cosh x}$

$\tanh x = \pm\dfrac{\sqrt{3}}{2}$

c Using $\cosh 2x = 2\cosh^2 x - 1$

$\cosh 2x = (2 \times 4) - 1$

$= 7$

4 **a** Using $\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - (-1)^2 = 1$

$\cosh^2 x = 2$

$\cosh x = \sqrt{2}$

cosh x cannot be negative.

b Using $\sinh 2x = 2\sinh x \cosh x$

$\sinh 2x = 2 \times (-1) \times \sqrt{2}$

$= -2\sqrt{2}$

c Using $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Alternatively use $\dfrac{\sinh 2x}{\cosh 2x} = \dfrac{\sinh 2x}{2\cosh^2 x - 1}$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{-1}{\sqrt{2}}$

$\tanh 2x = \dfrac{\left(-\frac{2}{\sqrt{2}}\right)}{1 + \left(\frac{1}{2}\right)}$

$= \dfrac{-2}{\sqrt{2}} \times \dfrac{2}{3}$

$= \dfrac{-4}{3\sqrt{2}} = -\dfrac{2\sqrt{2}}{3}$

5 a

$$3\sinh x + 4\cosh x = 4$$
$$\frac{3(e^x - e^{-x})}{2} + \frac{4(e^x + e^{-x})}{2} = 4$$
$$3e^x - 3e^{-x} + 4e^x + 4e^{-x} = 8$$
$$7e^x - 8 + e^{-x} = 0$$
$$7e^{2x} - 8e^x + 1 = 0$$
$$(7e^x - 1)(e^x - 1) = 0$$
$$e^x = \tfrac{1}{7} \text{ or } e^x = 1$$
$$x = \ln\left(\tfrac{1}{7}\right),\ x = 0$$

Multiply throughout by e^x.

Solve as a quadratic in e^x.

Note that $\ln\left(\tfrac{1}{7}\right) = \ln(7^{-1})$ $= -\ln 7$

b

$$7\sinh x - 5\cosh x = 1$$
$$\frac{7(e^x - e^{-x})}{2} - \frac{5(e^x + e^{-x})}{2} = 1$$
$$7e^x - 7e^{-x} - 5e^x - 5e^{-x} = 2$$
$$2e^x - 2 - 12e^{-x} = 0$$
$$e^x - 1 - 6e^{-x} = 0$$
$$e^{2x} - e^x - 6 = 0$$
$$(e^x - 3)(e^x + 2) = 0$$
$$e^x = 3$$
$$x = \ln 3$$

Multiply throughout by e^x.

$e^x = -2$ is not possible for real x.

c

$$30\cosh x = 15 + 26\sinh x$$
$$30\frac{(e^x + e^{-x})}{2} = 15 + 26\frac{(e^x - e^{-x})}{2}$$
$$15e^x + 15e^{-x} = 15 + 13e^x - 13e^{-x}$$
$$2e^x - 15 + 28e^{-x} = 0$$
$$2e^{2x} - 15e^x + 28 = 0$$
$$(2e^x - 7)(e^x - 4) = 0$$
$$e^x = \tfrac{7}{2},\ e^x = 4$$
$$x = \ln\left(\tfrac{7}{2}\right),\ x = \ln 4$$

Multiply throughout by e^x.

Solve as a quadratic in e^x.

5 d

$$13\sinh x - 7\cosh x + 1 = 0$$

$$13\frac{(e^{x} - e^{-x})}{2} - 7\frac{(e^{x} + e^{-x})}{2} + 1 = 0$$

$$13e^{x} - 13e^{-x} - 7e^{x} - 7e^{-x} + 2 = 0$$

$$6e^{x} + 2 - 20e^{-x} = 0$$

$$3e^{x} + 1 - 10e^{-x} = 0$$

Multiply throughout by e^{x}.

$$3e^{2x} + e^{x} - 10 = 0$$

$$(3e^{x} - 5)(e^{x} + 2) = 0$$

Solve as a quadratic in e^{x}.

$$e^{x} = \tfrac{5}{3}$$

$e^{x} = -2$ is not possible for real x.

$$x = \ln\left(\tfrac{5}{3}\right)$$

e

Use

$$\cosh 2x - 5\sinh x = 13$$

Using $\cosh 2x = 1 + 2\sinh^2 x$,

$$1 + 2\sinh^2 x - 5\sinh x = 13$$

$$2\sinh^2 x - 5\sinh x - 12 = 0$$

$$(2\sinh x + 3)(\sinh x - 4) = 0$$

$$\sinh x = -\frac{3}{2},\ \sinh x = 4$$

$$x = \operatorname{arsinh}\left(-\frac{3}{2}\right),\ x = \operatorname{arsinh} 4$$

$$x = \ln\left(-\frac{3}{2} + \sqrt{\frac{9}{4} + 1}\right)$$

$$= \ln\left(\frac{-3 + \sqrt{13}}{2}\right)$$

$$x = \ln\left(4 + \sqrt{16 + 1}\right)$$

$$= \ln\left(4 + \sqrt{17}\right)$$

5 f $3\sinh^2 x - 13\cosh x + 7 = 0$

Using $\cosh^2 x - \sinh^2 x = 1$,

$3(\cosh^2 x - 1) - 13\cosh x + 7) = 0$

$3\cosh^2 x - 13\cosh x + 4 = 0$

$(3\cosh - 1)(\cosh x - 4) = 0$

$\cosh x = \frac{1}{3}, \cosh x = 4$

$\cosh x = 4$

$x = \text{arcosh}\, 4, -\text{arcosh}\, 4$

$x = \ln\left(4 \pm \sqrt{4^2 - 1}\right)$

$= \ln\left(4 \pm \sqrt{15}\right)$

> $\cosh x \geq 1$, so $\cosh x = \frac{1}{3}$ is not

> Use $\text{arcosh}\, x = \ln\left(x + \sqrt{x^2 - 1}\right)$, but remember that $\ln\left(x - \sqrt{x^2 - 1}\right)$ is also a solution.

g $\sinh 2x - 7\sinh x = 0$

$2\sinh x \cosh x - 7\sin x = 0$

$\sinh x (2\cosh x - 7) = 0$

$\sinh x = 0, \cosh x = \frac{7}{2}$

$x = 0, x = \pm\, \text{arcosh}\left(\frac{7}{2}\right)$

$\text{arcosh}\left(\frac{7}{2}\right) = \ln\left(\frac{7}{2} + \sqrt{\frac{49}{4} - 1}\right)$

$= \ln\left(\frac{7 + \sqrt{45}}{2}\right)$

$= \ln\left(\frac{7 + 3\sqrt{5}}{2}\right)$

$x = 0, x = \ln\left(\frac{7 \pm 3\sqrt{5}}{2}\right)$

> Use $\text{arcosh}\, x = \ln\left(x + \sqrt{x^2 - 1}\right)$, but remember that $\ln\left(x - \sqrt{x^2 - 1}\right)$ is also a solution.

h $4\cosh x + 13e^{-x} = 11$

$4\frac{(e^x + e^{-x})}{2} + 13e^{-x} = 11$

$2e^x + 2e^{-x} + 13e^{-x} = 11$

$2e^x + 15e^{-x} - 11 = 0$

$2e^{2x} - 11e^x + 15 = 0$

$(2e^x - 5)(e^x - 3) = 0$

$e^x = \frac{5}{2}, e^x = 3$

$x = \ln\left(\frac{5}{2}\right), x = \ln 3$

> Multiply throughout by e^x.

> Solve as a quadratic in e^x.

5 i

$$2\tanh x = \cosh x$$

$$\frac{2\sinh}{\cosh x} = \cosh x$$

$$2\sinh x = \cosh^2 x$$

Using $\cosh^2 x - \sinh^2 x = 1$

$$2\sinh x = 1 + \sinh^2 x$$

$$\sinh^2 x - 2\sinh x + 1 = 0$$

$$(\sinh x - 1)^2 = 0$$

$$\sinh x = 1$$

$$x = \text{arsinh}\, 1$$

Use

$$x = \ln(1 + \sqrt{1^2 + 1})$$

$$= \ln(1 + \sqrt{2})$$

6 a

$$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$

$$= 2\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh 2x.$$

b Using the relation $\cosh 2x = 2\cosh^2 x - 1$, the equation becomes $2\cosh^2 x - 3\cosh x - 9 = 0$. Substituting $u = \cosh x$ into this equation allows us to clearly see a quadratic equation $2u^2 - 3u - 9 = 0$ which gives solutions $u = \dfrac{3 \pm \sqrt{9 + 72}}{4}$, we neglect the negative solution to avoid complex numbers.
i.e.
$u = 3.$
This means that our solutions for x are the values for
$x_1 = \text{ar}\cosh(3).$

Now recalling the expression $\text{ar}\cosh u = \ln\left(u \pm \sqrt{u^2 - 1}\right)$, we obtain the exact logarithm solutions

$$x = \ln\left(3 \pm 2\sqrt{2}\right).$$

7 Using the identity $\cosh^2 x - \sinh^2 x \equiv 1$, the equation in question becomes
$2(\cosh^2 x - 1) - 5\cosh x = 5$.
We simplify and use the substitution $u = \cosh x$ in order to obtain $2u^2 - 5u - 7 = 0$. This equation has solutions $u = \dfrac{5 \pm \sqrt{25+56}}{4}$, we neglect the negative solution to avoid complex numbers, thus $u = \frac{7}{2}$.
This means that our solutions for x are the values for
$x = \text{ar}\cosh\left(\frac{7}{2}\right)$.
Now recalling the expression $\text{ar}\cosh u = \ln\left(u \pm \sqrt{u^2 - 1}\right)$, we obtain the exact logarithm solutions
$x = \ln\left(\frac{1}{2}\left(7 \pm 3\sqrt{5}\right)\right)$.

8 $\text{sech}^2 x \equiv 1 + \tanh^2 x$ is not true, the correct identity is $\text{sech}^2 x \equiv 1 - \tanh^2 x$ by Osborn's rule.
He has split the fraction via denominator on the second line. This is not valid.
In mathematics we write this as
$\dfrac{a}{b+c} \neq \dfrac{a}{b} + \dfrac{a}{c}$.
The final mistake he made was assuming that taking the reciprocal of all terms preserves the value.
This is incorrect. For example $3 + 1 \neq \frac{1}{3} + 1$.
The correct proof is

$$\begin{aligned}\frac{1+\tanh^2 x}{1-\tanh^2 x} &\equiv \frac{2-\text{sech}^2 x}{\text{sech}^2 x}\\ &\equiv \frac{2}{\text{sech}^2 x} - 1\\ &\equiv 2\cosh^2 x - 1.\end{aligned}$$

9 a Recall the identity $R\cosh(x+a) \equiv R\cosh x \cosh a + R\sinh x \sinh a$.
From this we can set $R\cosh a = 10$ and $R\sinh a = 6$.
In order to find values for R and a:
$\dfrac{R\sinh a}{R\cosh a} = \tanh a = \dfrac{6}{10} = \dfrac{3}{5}$
giving

$$\begin{aligned}a &= \text{artanh}(0.6)\\ &\approx 0.693 (3 \text{ d.p.})\end{aligned}$$

We can now use the expression $R\cosh a = 10$.

To find $\cosh a$: $\text{sech}^2 a = 1 - \tanh^2 a = 1 - \left(\dfrac{3}{5}\right)^2 = \dfrac{16}{25}$

$\therefore \cosh^2 a = \dfrac{1}{\text{sech}^2 a} = \dfrac{25}{16}$ and so $\cosh a = \dfrac{5}{4}$

So $R = \dfrac{10}{\cosh a} = \dfrac{10}{\left(\frac{5}{4}\right)} = 8$

Thus giving us $10\cosh x + 6\sinh x = 8\cosh(x + 0.693)$.

9 b The minimum value occurs when $\cosh(x+0.693)$ is minimal. We know this occurs when $x+0.693=0$, by the graph of $\cosh x$. So the minimal value is $8\cosh 0=8$.
We could also find this answer by setting $y=8\cosh(x+0.693)$, finding the solution to $\frac{\mathrm{d}y}{\mathrm{d}x}=8\sinh(x+0.693)=0$ as $x=-0.693$ and substituting in, to get $y=8\cosh 0=8$.
The second derivative should be checked to be positive at this point in order to conclude that the stationary point is minimal, not maximal.

c $10\cosh x+6\sinh x=8\cosh(x+0.693)=11$.
We rearrange the equation to be $\cosh(x+0.693)=\frac{11}{8}$.
Set $u=x+0.693$ and now noting the symmetry about $u=0$ we obtain two solutions,
$u=\pm\operatorname{arcosh}\left(\frac{11}{8}\right)\approx\pm 0.841(3\text{ d.p.})$
This means our solutions for x are
$x_1=-1.534$,
$x_2=0.148$.